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NONUNIQUENESS OF COLLAPSE LOAD FOR A FRICTIONAL MATERIAL†

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Abstract—Nonuniqueness of collapse load for an isotropic frictional material with or without cohesion is considered. Examples are presented with solutions corresponding to two distinct collapse loads. A theorem is established which provides lower bounds for a nonempty although small class of plane strain problems.

INTRODUCTION

For an isotropic frictional material with or without cohesion,‡ a continuous plane deformation occurs by shearing without volume change along planes on which the Coulomb condition, $\tau = c - \sigma_N \tan\varphi$, holds. τ, σ_N, c and $\tan\varphi$ denote respectively magnitude of shear stress on the plane, normal stress on the plane, cohesion and coefficient of friction between adjacent surfaces of material points along the plane. In the stress boundary value problem of Fig. 1, T_x and v_y are specified components of surface traction and velocity respectively on the edges of a unit square. Obviously, homogeneously stressed deforming solutions are possible if slip planes§ lie in either the horizontal or vertical directions; it will be shown in Section 1 that these solutions correspond to distinct collapse loads‖ S. Furthermore, each of the solutions has a good physical interpretation for a certain value of the unspecified vertical normal stress.

This mechanism for distinct collapse loads can also manifest itself in less academically oriented problems. In Fig. 2 is shown an infinitely long grouser plate, which models a bulldozer track. V is dead load/unit length. Homogeneously stressed solutions involving horizontal plate translation with deformation occurring by shearing in the indicated shear zone are possible if slip planes lie in either the horizontal or vertical directions; it will also

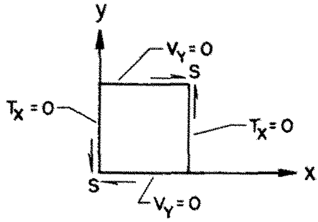

Fig. 1. Stress boundary value problem.

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‡ See Dais [1].

§ A plane upon which $\tau = c - \sigma_N \tan\varphi$ is termed a *slip plane*.

‖ In the theory of rigid perfectly plastic solids, limit analysis theorems guarantee the uniqueness of a collapse load for such a stress boundary value problem.

be shown in Section 1 that each of these solutions corresponds to a distinct collapse load H. Bekker [2] previously obtained the larger of these collapse loads by taking the slip plane in the horizontal direction.

A traditional approach† for obtaining failure loads for soil mechanics problems is to exhibit a critical‡ equilibrium stress field, solve for the load equilibrating the stress field and then simply term this load a "failure" load. Haythornthwaite [4] previously gave an example exhibiting distinct critical equilibrium stress fields which equilibrate distinct loads. Thus, Haythornthwaite showed that, for some problems at least, this traditional approach does not determine the "failure" load uniquely. The present approach, however, demands more of a load before it is termed a failure§ load. In addition to equilibrating a critical equilibrium stress field, a collapse load must be associated with a deforming solution in a theory involving material kinematics. This in fact is the usual meaning of the terms to analysts in plasticity theory. An example of distinct collapse loads as so defined has not been exhibited previously.

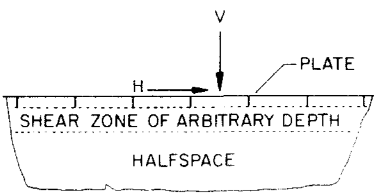

FIG. 2. Section of infinitely long grouser plate.

It is not presently known whether or not there exists a class of problems for which the collapse load is determined uniquely. Furthermore it is not known to what extent such a class of problems would intersect problems of technological interest, say those in Sokolovskii. It is known that at least some of the "failure" loads in Sokolovskii are indeed upper bounds on the collapse load as defined here; this follows from an upper bound theorem of Drucker [5] by using a procedure of Shield [6].

In Section 2 a lower bound collapse theorem is established for the class of stress boundary value problems for which any deformation is by definition restricted to occur in an x, y plane and $\bar{T}_z$, the zth component of surface traction is specified wherever n_z, the zth component of the unit normal vector, is not zero. Also, the zth component of body force must be zero. The statement of the theorem is: *Let* $\boldsymbol{\sigma}^*$ *be an equilibrium stress field satisfying stress boundary conditions and*

$$\sigma_1^* - \sigma_3^* < \frac{2\cos\phi}{1+\sin\phi}\left[-\frac{\bar{T}_z}{n_z}\sin\phi + c\cos\phi\right]$$

where $\sigma_1^* \geq \sigma_2^* \geq \sigma_3^*$ are principal components. *Then collapse cannot occur under loads equilibrating* $\boldsymbol{\sigma}^*$. In Section 2 a lower bound, which is likely to be unrealistically low, is found for the problem of the plane compressing of a right cylinder.

† See, for example, Sokolovskii [3].
‡ A state of stress at a point is said to be *critical* if $\tau = c - \sigma_N \tan\varphi$ on some plane through the point.
§ Or synonomously, "collapse" or "limit" load.

1. DISTINCT COLLAPSE LOADS

For the stress boundary value problem of Fig. 1, if a slip plane lies in the vertical direction, then obviously $\sigma_N = 0$ on the slip plane and the collapse load is given by $S = \tau = c$. If a slip plane lies in the horizontal direction, then it follows from the Mohr diagram of Fig. 3(a) that $S = c\cos^2\varphi/(1+\sin^2\varphi)$. For the problem of Fig. 2, if a slip plane lies in the horizontal direction, then obviously $H = c+V\tan\varphi$. If a slip plane lies in the vertical, then it follows from Fig. 3(b) that $H = (c+V\tan\varphi)/(2\tan^2\varphi+1)$.

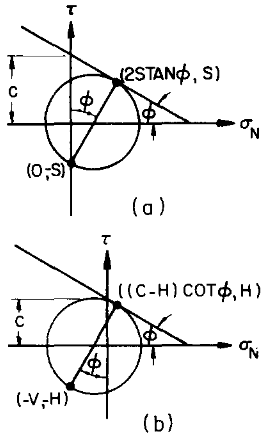

FIG. 3. Mohr diagrams.

2. A LOWER BOUND COLLAPSE THEOREM

For the plane straining of a body of frictional material in an x, y plane, the directions of σ_1 and σ_3 must lie in that plane, where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ denote principal stress components. Let $\dot{\varepsilon}_1 \geq \dot{\varepsilon}_2 \geq \dot{\varepsilon}_3$ be principal strain rate components. Then deformation occurs with no volume change and with $\dot{\varepsilon}_2 = 0$. The direction of $\dot{\varepsilon}_1$ lies in the x, y plane and is inclined at either $+\varphi/2$ or $-\varphi/2$ to the direction of σ_1. Take the zth component of body force to be zero. Equilibrium equations then require that σ_{zz} does not vary with z in a deforming region since the direction of σ_2 lies in the z direction there. Thus, in a deforming region, $\sigma_2 = \sigma_2(x, y) = \bar{T}_z(x, y)/n_z(x, y)$.

The rate of doing work will now be computed for a motion for which $\dot{\varepsilon}_1 = \alpha$. Let $\dot{\varepsilon}_{11}$ and $\dot{\varepsilon}_{33}$ denote respectively the normal strain rate component in the directions of σ_1 and σ_3 respectively. Then

$$\sigma_{ij}\dot{\varepsilon}_{ij} = \sigma_1\dot{\varepsilon}_{11}+\sigma_3\dot{\varepsilon}_{33}. \tag{1}$$

It follows from the Coulomb condition that

$$\sigma_1 = \frac{1-\sin\varphi}{1+\sin\varphi}\sigma_3+\frac{2c\cos\varphi}{1+\sin\varphi}, \tag{2}$$

and it can easily be shown that

$$\left.\begin{aligned}\dot{\varepsilon}_{11} &= \alpha\cos\varphi\\ \dot{\varepsilon}_{33} &= -\alpha\cos\varphi\end{aligned}\right\}. \tag{3}$$

From (1)–(3) it follows that

$$\sigma_{ij}\dot{\varepsilon}_{ij} = \frac{2\alpha \cos\varphi}{1+\sin\varphi}[-\sigma_3 \sin\varphi + c\cos\varphi]. \tag{4}$$

Since $\overline{T}_z/n_z = \sigma_{zz} \geq \sigma_3$ where $\alpha \neq 0$, there follows the relation

$$\sigma_{ij}\dot{\varepsilon}_{ij} \geq \frac{2\alpha \cos\varphi}{1+\sin\varphi}\left[-\frac{\overline{T}_z}{n_z}\sin\varphi + c\cos\varphi\right]. \tag{5}$$

Let $\boldsymbol{\sigma}^*$ denote a stress field which satisfies stress boundary conditions, the Coulomb limit condition, and the equilibrium equations. Let σ^*_{11} and σ^*_{33} denote respectively the normal stress components in the direction of $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_3$ respectively. Then

$$\sigma^*_{ij}\dot{\varepsilon}_{ij} = \alpha(\sigma^*_{11} - \sigma^*_{33}). \tag{6}$$

From relations (5) and (6) it follows that if

$$\sigma^*_{11} - \sigma^*_{33} < \frac{2\cos\varphi}{1+\sin\varphi}\left[-\frac{\overline{T}_z}{n_z}\sin\varphi + c\cos\varphi\right],$$

then

$$(\sigma_{ij} - \sigma^*_{ij})\dot{\varepsilon}_{ij} > 0 \tag{7}$$

wherever $\dot{\varepsilon} \neq 0$. Since $(\sigma^*_{11} - \sigma^*_{33}) \leq (\sigma^*_1 - \sigma^*_3)$, it follows that if

$$\sigma^*_1 - \sigma^*_3 < \frac{2\cos\varphi}{1+\sin\varphi}\left[\frac{\overline{T}_z}{n_z}\sin\varphi + c\cos\varphi\right], \tag{8}$$

then (7) holds wherever $\dot{\boldsymbol{\varepsilon}} \neq 0$. From a straightforward application of the principle of virtual work, it follows for the stress boundary value problem that collapse cannot occur under loads equilibrating $\boldsymbol{\sigma}^*$, provided that $\boldsymbol{\sigma}^*$ satisfies (8).

deJong [7] noted that if it could be shown that (7) holds, then a lower bound collapse theorem follows. The present theorem is thus an extension of deJong's work in the sense that a class of problems is specified for which (8) can be employed to insure that (7) holds.

Consider the problem of the compressing in plane strain of a right cylinder of unit end area and centroidally loaded between two smooth flat rigid end platens. Take the lateral surface of the cylinder to be traction free and then $\overline{T}_z/n_z$ can be taken to vanish in (8). If the y axis is taken perpendicular to the cylinder ends, then the stress field $[\sigma^*_x, \sigma^*_y, \tau^*_{xy}] = [0, -2c(1-\sin\varphi), 0]$ imposes equality in (8). It follows that a lower bound on collapse loads is given by $P^L = 2c(1-\sin\varphi)$. It follows, using Drucker's theorem, that $P^U = 2c\cos\varphi/(1-\sin\varphi)$ is an upper bound. The ratio of the bounds is $P^L/P^U = (1-\sin\varphi)^2/\cos\varphi$. If $\varphi = 30°$, then $P^L/P^U = 0{\cdot}29$.

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Абстракт—Исследуется неоднозначность нагрузки разрушения, для изотропного вязкостного материала, с учетом или без учета сцепления. Даются примеры с решениями, которые соответствуют двум различным нагрузкам разрушения. Предлагается теорема, определяющая нижные пределы для непустого, но даже малого класса задач, касающихся плоской деформации.